Monads and Theories
or
What the Hell is Algebra?

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Outline

Introduction

Algebraic Theories

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Monads
What?

Monads
In antichronological order:

- Functional programming (e.g. Haskell)
- Denotational semantics (Moggi)
- Category theory (algebraic theories)

Universal algebra
Models of algebraic theories in

- **Set** (Birkhoff)
- Categories with finite products (Lawvere)
- Enriched categories with finite cotensors (Kelly & Power)
Why?

**Programming**
The connection between monads and algebra is not often explained.

**Semantics**
Some have advocated using algebraic (Lawvere) theories in semantics instead of monads — there are tensors and sums of theories (Power, Plotkin, Hyland).

**Usefulness**
I may want to talk about logics as 2-categories with ‘algebraic’ structure — the enriched version of the theory could be useful. Also, it’s pretty nifty.
How?

- We will show in detail how a standard algebraic theory gives rise to a finitary monad on \textbf{Set}.
- Everything generalizes neatly to enriched categories and infinite arities.
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The Classical Approach

An algebraic theory $T$ is given by

- a set $S$ of *operations* together with an *arity* function $\text{ar} : S \to \mathbb{N}$
- a set $E$ of *equations* between terms formed from variables and operations of $S$.

A model of $T$ or a $T$-algebra is given by

- a set $A$ together with functions $[f]_A : \text{Ar}(f) \to A$ for each $f \in S$ (i.e. an $S$-algebra), such that
- the interpretations of the terms equated by $E$ are equal.
What Does That Mean?

- Terms are elements of a free $S$-algebra (also called a term algebra, Herbrand universe, etc.).
- Think of a set $A$ as a set of variables $\{x_1^A, x_2^A, \ldots\}$.
- Then the free $S$-algebra $S^*A$ on $A$ is defined by induction:

\[
S^0A = A \\
S^{n+1}A = A \cup \bigcup_{f \in S} \{f\} \times (S^nA)^{\text{ar}(f)}
\]

The superscript in $S^n$ is the maximum depth of a term.
- The set $S^*A$ is then $\bigcup_{n \in \mathbb{N}} S^nA$. 
A free $S$-algebra is indeed an $S$-algebra:

\[ [f] : \langle t_1, \ldots, t_{\text{ar}(f)} \rangle \mapsto f(t_1, \ldots, t_{\text{ar}(f)}) \]

For example, given $S = \{ \bar{0}, \text{succ} \}$, with $\text{ar}(\bar{0}) = 0$ and $\text{ar}(\text{succ}) = 1$, $S^*A$ looks like:

\[
\bar{0}, x_1^A, x_2^A, \ldots \\
\text{succ}(x_2^A), \text{succ}(\bar{0}), \ldots \\
\text{succ}(\text{succ}(x_1^A)), \text{succ}(\text{succ}(\bar{0})), \ldots \\
\vdots
\]
Equations

$[-]_A$ lifts uniquely to $S^*A$:

$$[f(t_1, \ldots, \text{tar}(f))] S^*A = [f]_A \circ \langle [[t_1] S^*A, \ldots, [\text{tar}(f)] S^*A] \rangle$$

- Equations have two sides, left and right, so an equation is a pair $\langle l, r \rangle$ of terms.
- Given $T = \langle S, E \rangle$, an $S$-algebra $A$ is a $T$-algebra if $[[l]] S^*A = [[r]] S^*A$ for each equation $\langle l, r \rangle \in E$.
- The free $T$-algebra on $A$ is $S^*A/\sim$, where $t \sim u$ if there is some $\langle l, r \rangle \in E$ such that $[[l]] = t$ and $[[r]] = u$. 
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What Are Monads Good For?

CS
Computer scientists often encounter the following setup:
- A category \( C \) of ‘pure’ programs (the semantics of a language).
- A monad \( T \) on \( C \), where \( TA \) is the type of (effectful) computations returning an \( A \).

Then the semantics of the language plus effects is the Kleisli category \( C_T \).

Maths
Mathematicians think of monads as follows:
- \( C \) is a universe of objects.
- A monad \( T \) on \( C \) yields free \( T \)-structures \( TA \).

The interesting thing then is the category \( C^T \) of \( T \)-algebras.
Where Do Monads Come From?

In the wild, monads typically arise from pairs of adjoint functors. This is where we have two functors like so:

\[
\begin{array}{c}
C \\ F \\
\downarrow \\
\downarrow \\
U \\ D
\end{array}
\]

and a natural isomorphism

\[\text{hom}_D(FA, B) \cong \text{hom}_C(A, UB)\]

Example: take \(C = \text{Set}\) and \(D = \text{Grp}\). \(U\) is ‘underlying set’ and \(F\) is ‘free group’. The isomorphism says that any homomorphism out of a free group is uniquely determined by its values at the generators.

This is what *free* means!
The composite $UF$ takes a set $A$ to the underlying set of the free group on $A$ — that is, the set of all formal terms of group theory over $A$.

Feeding the identities into the isomorphism we get:

$\text{hom}_D(FA, FA) \cong \text{hom}_C(A, UFA)$

$1_{FA} \mapsto \eta_A$

and

$\text{hom}_C(UB, UB) \cong \text{hom}_D(FUB, B)$

$1_{UB} \mapsto \epsilon_B$
If $UFA$ is the free group on $A$, then

- $\eta_A$ simply ‘includes the generators’ — it sends the elements of $A$ to themselves in $UFA$.
- $UFUFA$ is the ‘free group on the free group’ — it consists of formal terms whose ‘variables’ are formal terms in $A$.
- $\mu_A = (U\epsilon F)_A : UFUFA \to UFA$ takes these terms in terms and ‘multiplies them out’ to get terms in $A$. 
Examples

Free Monoids (Lists)

- $F$ is ‘free monoid’, $U$ is ‘underlying set’. $UFA$ is $A^*$, the set of lists of elements of $A$.
- $\eta_A : A \rightarrow A^*$
  
  \[
  a \mapsto [a]
  \]

  (singleton list)
- $\mu_A : A^{**} \rightarrow A^*$
  
  \[
  [[a_1, \ldots, a_n], \ldots, [z_1, \ldots, z_m]] \mapsto [a_1, \ldots, a_n, \ldots, z_1, \ldots, z_m]
  \]

  (concatenate)
Examples

Free Modules (Polynomials)

Let $R$ be a ring.

- Functors

  \[
  \begin{array}{ccc}
  \text{Set} & \overset{\mathcal{F}_G}{\longrightarrow} & \text{Ab} \\
  \mathcal{U}_G & \overset{\mathcal{F}_M}{\longleftarrow} & R\text{-Mod}
  \end{array}
  \]

  \[UFA = U_G U_M F^M F^G.\]
  
  $A$ is $R[A]$ — polynomials in $A$ with coefficients in $R$.

- $\eta_A : A \to R[A]$

  \[x \mapsto x\]

- $\mu_A : R[R[A]] \to R[A]$

  [Multiplies out polynomials in polynomials]
So What’s A Monad?

A *monad* on a category $\mathbf{C}$ is a functor $T : \mathbf{C} \to \mathbf{C}$ together with natural transformations $\eta : 1 \to T$ and $\mu : T^2 \to T$, satisfying:

\[
\begin{array}{ccc}
T & \xrightarrow{T \eta} & T^2 \\
\downarrow \eta T & \cong & \downarrow \mu \\
T^2 & \xrightarrow{\mu} & T
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{ccc}
T^3 & \xrightarrow{T \mu} & T^2 \\
\downarrow \mu T & \cong & \downarrow \mu \\
T^2 & \xrightarrow{\mu} & T
\end{array}
\]

If $F \dashv U$, then $\langle UF, \eta, U\epsilon F \rangle$ is always a monad.

Monads on $\mathbf{C}$ are exactly monoid objects in the monoidal category $\langle [\mathbf{C}, \mathbf{C}], \circ, 1_\mathbf{C} \rangle$. 
Monads From Theories

Given a signature $S$, its *structure functor* $F_S$ is defined as:

$$F_S X = \sum_{f \in S} X^\text{ar}(f)$$

An $F_S$-algebra is a map (or action) $a : F_S A \to A$.

$S$-algebras are $F_S$-algebras:
Example

Let $P = \{ \overline{0}, \text{succ} \}$ as before. Then we have

$$F_P X = \sum_{f \in P} X^{\text{ar}(f)} = X^0 + X^1 \cong 1 + X$$

A $P$-algebra is thus given by an action $p : 1 + X \rightarrow X$.

The initial $P$-algebra is $\mathbb{N}$. (Remember Arthur’s talk?)
Equivalently, we can group operations by arity. For a signature \( \langle S, \text{ar} \rangle \), write \( S \) again for \( \text{ar}^{-1} : \mathbb{N} \to \text{Set} \). So

\[
Sn = \{ f \in S \mid \text{ar}(f) = n \}
\]

Now \( F_S \) involves a sum over arities:

\[
F_S X = \sum_{n \in \mathbb{N}} Sn \times X^n
\]

An \( F_S \)-algebra is the same as before.
Monads From Theories III

We want to take free $S$-algebras. These are defined as before:

\[
S^0 X = X \\
S^{n+1} X = X + F_S(S^n X) \\
S^* X = \text{colim}_n S^n X
\]

This makes $S^* A$ the initial algebra for the functor $X \mapsto F_S X + A$, the colimit of

\[
0 \overset{!}{\longrightarrow} F_S 0 + A \overset{F_S! + A}{\longrightarrow} F_S(F_S 0 + A) + A \longrightarrow \cdots
\]

If $F_S$ is nice (preserves countable coproducts), we have

\[
S^* \cong 1 + F_S + F_S^2 + \cdots = \bigsqcup_n F_S^n
\]
This makes \( S^* \) the \textit{free monad} on \( F_S \):

\begin{itemize}
  \item \( \eta : 1 \rightarrow S^* \) is simply the injection into the first summand (inclusion of the generators).
  \item \( \mu : S^*S^* \rightarrow S^* \) is

\[
\bigsqcup_n F^n_S \circ \bigsqcup_m F^m_S \xrightarrow{\sim} \bigsqcup_{m,n}(F^n_S \circ F^m_S) \xrightarrow{\sim} \bigsqcup_k F^k_S
\]

an isomorphism!
\end{itemize}

\textbf{Why?}

Informally, \( S^*S^*A \) consists of formal terms whose leaves are again formal terms in \( A \). Given such a term, suppose the two layers have depth \( n \) and \( m \) — then the term is a term in \( A \) of depth \( n + m \).
Simple Example

Consider the simplest non-trivial signature: let $S_0 = \{ f \}$, and $S_n = \emptyset$ for $n > 0$. Now

$$F_S : X \mapsto \sum_{n \in \mathbb{N}} S_n \times X^n$$

$$\implies 1 \times X^0$$

$$\implies 1$$

So $F_S$ is the constant functor at the one-element set. The free monad on it is

$$S^* A = 1 + A$$

In Haskell this is called the Maybe monad. The operation $f$ plays the role of the Nothing constructor.
Some facts:

- The arities of a signature are natural numbers.
- Naturals are isomorphism classes of finite sets (addition is coproduct, product is product).
- Any set is the union of its finite subsets.
- All of the functors we have constructed are finitary — they preserve directed unions of finite sets.

This means that we need only consider the images of the functors $F_S$ and $S^*$ at finite sets, since the above shows

$$S^* A \cong S^* (\bigcup P_f(A)) \cong \bigcup S^*(P_f(A))$$

Now $S^* n$ is the free $S$-algebra on $n$ generators (unique up to isomorphism) — or the set of $S$-terms of arity $n$. 
Equations

Let $T = \langle S, E \rangle$ be a theory.

- Define the arity of an equation to be the least $n$ such that neither side contains more than $n$ distinct variables.
- Now define a set of equations to be a function $E : \mathbb{N} \to \textbf{Set}$ sending $n$ to a set of (abstract) operations of arity $n$.
- Interpreting an equation’s left and right sides as $S$-terms yields two functions for each $n$:

$$En \xrightarrow{l_n} S^* n \xleftarrow{r_n}$$

For example, the theory of groups will involve $E3 = \{\text{assoc}\}$, where

$$l_3(\text{assoc}) = x_1 \cdot (x_2 \cdot x_3)$$
$$r_3(\text{assoc}) = (x_1 \cdot x_2) \cdot x_3$$
Equations II

- Just as with a signature, there is a free monad on $E$.
- The maps $l_n, r_n : E^n \rightarrow S^* n$ lift uniquely (because of freeness) to monad maps

$$E^* \xrightarrow{l^*} S^*$$

$$E^* \xrightarrow{r^*} S^*$$

- The monad $T^*$ corresponding to $T$ is the coequalizer of these, so for each $n$ we have

$$E^* n \xrightarrow{l_n^*} S^* n \xrightarrow{r_n^*} T^* n$$
Algebras

$T$-algebras for a monad $T$ should interact properly with the extra structure on $T$. A $T$-algebra is an arrow $a : TA \to A$ as before, such that

\begin{align*}
T^2 A &\xrightarrow{T a} TA \\
\mu_A &\downarrow \\
TA &\xrightarrow{a} A
\end{align*}

\begin{align*}
A &\xrightarrow{\eta_A} TA \\
1 &\downarrow \\
A &\xrightarrow{a} A
\end{align*}

A homomorphism of $T$-algebras is a morphism of the underlying objects that respects the algebra structure:

\begin{align*}
TB &\xrightarrow{T f} TA \\
b &\downarrow \\
B &\xrightarrow{f} A
\end{align*}

$T$-algebras form a category called $\mathbf{C}^T$ or $T$-$\mathbf{Alg}$. 
The Kleisli Category

If $T$ is thought of as representing a particular kind of side-effect, then the relevant arrows are *Kleisli arrows* — an effectful program from $A$ to $B$ is modelled as an arrow $f : A \to TB$.

The Kleisli category $C_T$ (or $\text{Kl}(T)$) of $T$ is defined to have

- **Objects**: those of $C$.
- **Arrows** $f : A \rightsquigarrow B$ in $C_T$ are arrows $f : A \to TB$ in $C$.
- **Composition**: given $A \rightsquigarrow B \rightsquigarrow C$, their composite is

$$A \xrightarrow{f} TB \xrightarrow{Tg} T^2C \xrightarrow{\mu_C} TC$$

$\mu_C \circ Tg$ is the ‘unique lifting’ of $g$ to the free $T$-algebra on its domain.
The Kleisli category $\mathbf{C}_T$ is equivalent to the subcategory of $T$-$\mathbf{Alg}$ consisting of the free algebras $\mu_A : T^2A \to TA$.

- Objects $A$ of $\mathbf{C}_T$ (i.e. objects of $\mathbf{C}$) uniquely determine free algebras $TA$ and actions $\mu_A$.
- Kleisli arrows $f : A \sim B$ lift uniquely to arrows $\mu_B \circ Tf$.
- These are morphisms of free algebras by virtue of

$$
\begin{array}{cccc}
T^2 A & \xrightarrow{T^2f} & T^3 B & \xrightarrow{T\mu_B} & T^2 B \\
\mu_A & & \mu_{TB} & & \mu_B \\
TA & \xrightarrow{Tf} & T^2 B & \xrightarrow{\mu_B} & TB
\end{array}
$$