Global Portfolio Selection: The Case of an Investor with Mean-Variance Preferences

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DECLARATION

I hereby declare that this project is entirely my own work and that it has not been submitted as an exercise for a degree at this or any other university.

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Abstract

The aim of this paper is to assess the performance of the Markowitz mean-variance framework over a thirty year time frame and address the question of; How should an investor optimally allocate their capital?. The effect of risk reduction by incorporating a Bayes-Stein estimator is also investigated. The performance of the framework is concluded by the out-of-sample performance of the mean-variance portfolios as well as 4 other key portfolios.

The paper also describes an application implemented to perform a similar analysis on any data set and to view the results graphically.
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Chapter 1

Introduction

1.1 Overview

Academics and investors have always had an interest in stock market prediction because of the traditionally high level of returns associated with them and also because of the many different statistical and behavioural aspects that make up the determinants of stock price movements.

Of these statistical aspects the idea of risk is of keen interest to any investor as it is perhaps the most important factor when it comes to stock market investing. Investors want to diversify and abide by the old adage "don’t put all your eggs in one basket". Investors diversify their risk by allocating their capital across a portfolio of different assets. These assets may be stocks, bonds, derivatives or real estate essentially any asset class.

Within financial markets there are many ways to handle risk. Traditionally fundamental analysis was seen as the best way of identifying the direction a stock price would move and an investor would mitigate the risk be either buying, selling or holding the stock. This is perhaps the simplest and well
known approach to offsetting the potential capital loss risk. This method consists of examining the chosen company (e.g. by using accounting data) as well as the industry/sector it operates in and the performance of the economy supporting it. Then concluding the direction the stock will move and pursuing an investment strategy to match this forecast. There are major flaws and drawbacks in using this method and it is almost impossible to predict the consequences of market fluctuations on stock prices by just using fundamental analysis. This method also focuses on individual assets and not a portfolio of assets.

An alternative tool that gained prominence was the idea of diversifying risk by allocating capital across a portfolio of different assets. There are many different approaches proposed on how to address this problem of portfolio selection in order to reduce risk. In this paper the standard mean-variance approach is applied to address this problem. It assumes that the investor’s utility function is that of only risk and return. For a specified target level of return the mean-variance framework will yield the portfolio weights that give the lowest variance for a given set of risky assets. The risky assets in this paper consist of ten equity indices discussed in Chapter 4.

Modern Portfolio Theory (MPT) is the literature behind the mean-variance (mv) framework. MPT first emerged in the 50’s and 60’s with the ‘father’ of modern portfolio theory Harry Markowitz through his 1952 paper ‘Portfolio Selection’ and 1959 book ‘Portfolio Selection: Efficient Diversification of Investments’ [1][2]. Although the opposition and arguments with this theory are steadily growing it is still the principal foundation for financial theory being taught in the academic world and also the basis of many portfolio manager’s
investment strategies.

MPT utilises sample estimates of expected returns and variances based on empirical data allowing the formulation of a so called 'efficient frontier'. The efficient frontier is the set of all optimal mv portfolios for different levels of target return. From where an investor can choose a preferred portfolio that for their target return minimises the risk(measured by the variance of returns). Conversely given an investor’s target minimum risk maximises their return. This mv framework will be discussed in greater detail in Chapter 2.

One of the main arguments associated with this framework is the problem of estimation error in the inputted mv parameters (Michaud (1989) [3]. As sample empirical data is used to calculate the expected return and variance of optimal portfolios and then treating these as true certain values. One approach put forward to reduce this estimation error is the use of a Bayes-Stein(bs) estimator as advocated by Jorian (1985) [4]. Jorion proposed a bs estimator that smooths the sample estimate of expected return towards a global estimate. This approach as well as the global estimate will be explained in more detail in Chapter 2.

Ultimately this paper is concerned with analysing the performance of the Markowitz mv framework by using empirical data from ten major stock market indices over a thirty year period and addressing the investors allocation of capital question. The effect of using a bs estimator to address one of the biggest criticisms of the mv framework is also tested. Assessing the out-of-sample performance of the mean-variance portfolios over a moving window is how we are going to conclude the performance of the mv framework and the bs estimator.
1.2 Structure

Chapter 2 and 3 will delve into the relevant academic literature associated with MPT, examining the advantages and criticisms of the theory as well as introducing the basic portfolio notation and presenting the formulae used for the mv optimisation process.

The Data Analysis chapter will discuss the data used, its source and the reasons for the specific timeframe chosen.

Chapter 5 and 6 are the integral chapters in this paper. Chapter 5 presents the results of our optimisation simulations as well as constructing the efficient frontiers for the data set. Chapter 5 explains how the application is implemented as well as some of the algorithms behind the functionality of the application also describing user interaction with the event-driven Matlab GUI.

Chapter 7 then provides the conclusion from our analysis and optimisation results as well as future extensions and improvements that would be of benefit to this paper and application.
Chapter 2

Literature Review

2.1 Diversification

The idea of diversifying is not a new concept in fact it was prevalent, although not fully understood, long before the establishment of modern finance theory. Investors always abided by the old adage 'don’t put all your eggs in one basket' (5). Simply put diversification is a way or reducing an investors risk by investing their capital across a variety of assets not just putting all their capital in one specific asset. Risk in this case is measured by the variance of asset price returns.

Take the example of investing in a single stock, it would not be uncommon for a single stock to increase or decrease in value by 40% annually whereas if an investor had a portfolio of stocks it would be extremely unlikely that the aggregate or portfolio return would go up or down by 40%. The latter is an example of diversification. The less correlated these assets are the greater the diversification benefits. This concept of asset return risk can be split into two components:
- Specific risk, often referred to as indiosyncratic risk, is the risk of individual assets. Specific risk is the possibility that the price of an asset may decline due to an event that could affect that particular asset but not the market as a whole. Specific risk can be reduced through diversification [6].

- Systematic Risk, often referred to as market risk or undiversifiable risk, is risk that cannot be diversified away. One simple example would be a world war as this would affect global stock prices and cannot be diversified away by allocating across a portfolio [7].

### 2.2 Modern Portfolio Theory

Before the introduction of Modern Portfolio Theory (MPT) the concept of risk and return was treated in a casual manner with no quantifiable relationship. Investors were risk averse and intuitively invested in multiple securities (diversified) in the hope that a loss in one security would be offset by a gain in the others. Investors did not take into account the movements of assets in respect to each other (their covariance) as a way of further diversifying, they understood risk but had no way of quantifying it.

An American economist by the name of Harry Markowitz was the first one to present a portfolio selection framework based on these diversification concepts in his 1952 paper Portfolio Selection and subsequent 1959 book. He provided an answer to the question; How should an investor allocate their capital among the possible investment choices? Markowitz for the first time formulated the portfolio problem as a choice between the mean and variance (mv) of a portfolio of asset returns. He pur-
posed that investors should consider risk and return together and determine
the allocation of their funds among investment alternatives on the basis of
the trade-off between them [8]. These papers have formed the foundations
of what is now popularly referred to as Modern Portfolio Theory (MPT).

He used the statistical measure of covariance when formulating the optimisation
problem for reducing risk. Which based the idea of a portfolio's riskiness
on the covariance of the assets in the portfolio and not just the average risk-
iness of each individual asset. Markowitz showed that the overall risk of a
portfolio could be lowered by the addition of an asset with a low, positive
or negative covariance to other assets in the portfolio without reducing the
overall expected return of the portfolio [9].

The covariance or correlation in portfolio theory is a measure of how assets
move together, so in essence investing all your wealth in highly correlated
assets is far riskier than investing in uncorrelated assets or assets that move
in opposite directions. That is if one asset is declining assets that are highly
correlated will also be declining, leading to an investor's entire portfolio de-
clining amplifying their losses. This concept was completely new to financial
analysis as up to that point investors where only concerned with individual
assets and fundamental analysis, essentially picking winners - assets that offer
the highest expected return.

MPT formulated an investor's financial decision-making process as an opti-
misation problem. The Markowitz mean-variance framework offered optimal
portfolios which for a specified target return minimised the variance. A risk
averse investor should choose the optimal portfolio that matches their level
of risk aversion as this portfolio will minimise the variance of their returns.
All other portfolios that offer the same return are ‘inefficient’ as they have a larger variance thus higher risk. This led to the creation of an efficient frontier Fig 2.1. The efficient frontier is a curve that shows all the optimal portfolios from the feasible set (all portfolios that can be constructed from the available assets) depending on the investors risk appetite. Portfolios that lie below the curve are said to be ‘inefficient’ as for that target level of return there are a set of weights that will yield a lower portfolio variance.

The inputs to this portfolio optimisation framework described by Markowitz are the expected returns, volatility (variance of returns) and correlation estimates as well as the constraints put on portfolio composition. This allows an investor to find the weights that minimise the overall portfolio variance for a target level of return. Thus systematically eliminating idiosyncratic risk by creating an optimal diversified portfolio. This process is shown in Fig 2.2 with the optimisation formulae themselves being presented in Chapter 3. The ultimate output of the process is the efficient frontier.
2.3 Portfolio Theory Criticism

2.3.1 Estimation Error

The most cited and common criticism of the mean-variance framework is the difficulty to make accurate estimates of the many inputs required in the MPT process. The problem is not with the Markowitz theory and framework but the uncertainty of the optimisation inputs. The calculation for expected return relies on using sample estimates of empirical data. If the time series for the empirical data was infinitely long then there would be no estimation error, but in reality time series of asset prices are not long therefore estimation error occurs [12]. Findings show that small changes in input parameters imply large changes in the optimised portfolio composition and considerable
modification of the mv efficient frontiers shape [13]. These estimation errors in the inputs can significantly affect the resulting portfolio weights where extreme weights can be produced for some of the assets.

Literature shows that in practical applications equally weighted portfolios (same weighting on each asset in portfolio) out-of-sample performance is far greater than mv efficient portfolios [14]. This is not necessarily a sign that the Markowitz theory is flawed but rather when applied in practice it has to be modified to achieve reliability and robustness to deal with estimation error in the inputs [15].

Traditionally the majority of research on Markowitz mv optimisation has focused on the in-sample performance of the portfolios thus ignoring the uncertainty in the estimation error or the out-of-sample performance, an example of this is Black and Litterman (1992) [16]. This present paper optimises both in-sample and out-of-sample to get a better view of the performance of the mv framework.

### 2.3.2 Normal Distribution

Following on from estimation error, another criticism of MPT is the assumption that stock price movements follow a Gaussian normal distribution. Essentially implying that 68% of all price changes would be within one standard deviation of the mean, 95% within two standard deviations and 98% within three. In reality economists have been well aware that the variation of stock prices do not match this normal distribution and that stock prices can increase or decrease by very large amounts fairly often [17]. The recent 2008 financial crisis, the 1987 stock market crash or the several thousand per cent
increase in short-term interest rates of the 90’s are examples of this-known as 'outliers' or 'black swan' events [18]. Assuming the normal distribution and using variance as a risk measure underestimates and is insensitive to the occurrence of these extreme events and thus portfolio risk can be underestimated. It has been shown that to an investor these events are of most interest as they can have the largest impact on an investors wealth thus other distributions such as student-t and skewed have been proposed to account for this non-normality and skewness of prices [19].

2.3.3 The Efficient Market Hypothesis

The widely held belief in the Efficient Market Hypothesis which states that financial markets are informationally efficient and that all available information is represented in the current market stock price is in complete contradiction to MPT. As a consequence of this belief it follows that an investor cannot consistently achieve returns in excess of the average market return on a risk adjusted basis given the information available at the time [20]. It implies no stocks are undervalued as financial analysts utilise all available information to ensure appropriate prices. What are the implications of an efficient market for portfolio management?

Investors who believe in this hypothesis would practice passive investment (investing in a broad based index replicating the market) and regard active investment such as technical analysis as futile. If all information is reflected in the current price then it is impossible to beat the market in the long run. In practice it has been shown that prices cannot perfectly reflect the information available since information is costly and those who spent resources to obtain it would receive no compensation so why would they [21].
Warren Buffet also has shown that he consistently beats the market in the long run and as he famously said 'I'd be a bum on the street with a tin cup if the market was always efficient' [22].

2.4 Bayes-Stein Estimator

As mentioned earlier an optimal portfolio is derived using mv optimisation where the inputs are the expected returns, variances and covariances. But these are simply got form the ex post sample values and nothing is said of the uncertainty of these inputs. A rational investor should note this uncertainty when forming expectations about future portfolio performance and seek to consider estimators that are less subject to this estimation error associated with the classical sample mean [23]. It is crucial to account for the impact on portfolio analysis these errors in estimating expected returns have- as they are sample estimates based on empirical data.

The investor can increase the accuracy of the estimate by using longer time series. If the length of the time series goes to infinity both the expected returns and the covariance matrix can be estimated exactly [24]. In reality time series are not long enough and input parameters cannot be estimated with precision. This estimation error is the reason for the poor out of sample performance of mv portfolios and why so many academics have focused their studies on in-sample performance [25].

One approach to reduce this estimation error is the bs estimator. Proposed by Jorion(1985) as an alternative method of estimating expected returns. Essentially each asset mean is 'shrunk' towards a common value. Shrinkage
estimators like the bs estimator have been shown to account for this parameter uncertainty in portfolio optimisation [26]. Jorion proposes this common value to be the global minimum variance portfolio (gmv) which can be estimated with relative precision as it is solely constructed using the covariance matrix of stock returns discussed in Section 3.4.1. Using a bs estimator to ‘shrink’ the sample mean towards the gmv portfolio reduces the estimation error associated with expected returns. This is the central idea behind Stein’s estimator and the formulae for this ‘shrinkage factor’ estimator are presented in equation 3.24.
Chapter 3

Portfolio Theory

3.1 Basic Portfolio Notation

The returns of \( N \) equity indices is assumed to follow a multivariate normal distribution (more than one random normal distributed variable) denoted

\[
X \sim N(\mu, \Sigma)
\]  

(3.1)

where \( \mu \) is a vector of mean index returns denoted \( \mu = (\mu_1, \mu_2, \ldots, \mu_N) \), and \( \Sigma \) is the \( N \times N \) variance-covariance matrix of index returns given by

\[
\Sigma = \begin{bmatrix}
\sigma_{11} & \cdots & \sigma_{1N} \\
\vdots & \ddots & \vdots \\
\sigma_{N1} & \cdots & \sigma_{NN}
\end{bmatrix}
\]

where \( \sigma_{ij} \) is the covariance of index \( i \) and index \( j \) such that \( \sigma_{ii} = \sigma_i^2 \) (the variance of index \( i \)). The covariance between between index \( i \) and \( j \) is the following \( \sigma_{ij} = \rho_{ij} \sigma_i \sigma_j \) where \( \rho_{ij} \) is the correlation of index \( i \) and index \( j \) [27].
Let \( w = (w_1, w_2, \ldots, w_N) \) be a vector of portfolio weights where \( w_1 \) is the percentage of capital allocated to index 1. This paper permits short selling (see appendix 1) so the weights \( w \) can take on negative values. Allowing short selling results in a closed form solution for the weights. The overall mean portfolio return is a weighted sum of the mean index returns given by

$$
\mu_P = w^T \mu
$$  \hspace{1cm} (3.2)

The overall portfolio variance is a weighted sum of the index variances and covariance given by the quadratic product of the portfolio weight vector and the variance-covariance matrix

$$
\sigma_P^2 = w \Sigma w^T
$$  \hspace{1cm} (3.3)

### 3.2 Naive 1/N Diversification

An equal weighted portfolio is defined such that \( w_i = N \) and \( i = 1, \ldots, N \), where the investor allocates their capital evenly across each index, in this case the overall mean portfolio return is given by

$$
\mu_P = \sum_{i=1}^{N} \frac{1}{N} \mu_i
$$  \hspace{1cm} (3.4)

with portfolio variance

$$
\sigma_P^2 = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{N} \frac{1}{N} \sigma_{ij}
$$  \hspace{1cm} (3.5)

$$
= \frac{1}{N} \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 + \frac{N-1}{N} \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j \neq 1}^{N} \sigma_{ij}
$$  \hspace{1cm} (3.6)

$$
= \bar{\sigma}_{ij} + \frac{1}{N} \left( \bar{\sigma}_i^2 - \bar{\sigma}_{ij} \right)
$$  \hspace{1cm} (3.7)
[29] given:

\[ \sigma_i^2 = \frac{1}{N} \sum_{i=1}^{N} \sigma_{ii} \sigma_{ij} \]  

\[ \sigma_{ij}^2 = \frac{1}{(N)(N-1)} \sum_{i=1}^{N} \sum_{j \neq 1}^{N} \]  

where \( \sigma_i^2 \) represents the average variance across the N assets and \( \sigma_{ij} \) denotes the average covariance. Given such a formulation the portfolio variance approaches the average covariance as \( N \) increases.

\[ \lim_{N \to \infty} \sigma_{ij} + \frac{1}{N} (\sigma_i^2 - \sigma_{ij}) = \sigma_{ij} \]  

(3.10)

The lower the covariance \( (< 0) \) between the investments and the larger the number of investments (up to a point) the greater the potential for risk reduction. However, the \( 1/N \) weighting is a naive diversification approach and theoretically further risk reduction can be achieved by optimising over the weights \( w \). The Markowitz optimisation framework allows an investor to find the weights \( w \) to minimise overall portfolio variance for a target level of return.

### 3.3 Optimal Diversification Using Mean-Variance Optimisation

The mean-variance framework can be used to determine the weight vector that minimises overall portfolio variance for a target level of return. This results in an optimisation problem of the form

\[ \min_w w^T \Sigma w \]  

(3.11)
subject to the constraints

$$\mu_0 = w^T \mu$$  \hspace{1cm} (3.12)

$$w^T l = 1, l^T = [1, 1, ..., 1]$$  \hspace{1cm} (3.13)

The above is the classical mean-variance optimisation problem known as the risk minimisation formulation as it minimises the variance for the specified target return ($\mu_0$). The solution for $w$ is given by

$$w = g + h \mu_0$$  \hspace{1cm} (3.14)

where;

$$g = \frac{1}{ac - b^2} \sum_{t=1}^{T} \left[ cl - b \mu \right]$$ \hspace{1cm} (3.15)

$$h = \frac{1}{ac - b^2} \sum_{t=1}^{T} \left[ a \mu - bl \right]$$ \hspace{1cm} (3.16)

and;

$$a = l^T \sum_{t=1}^{T} \mu : b = l^T \sum_{t=1}^{T} \mu : c = \mu^T \sum_{t=1}^{T} \mu$$ \hspace{1cm} (3.17)

Equation 3.14 is an analytical solution that gives the weights for each of the indices in the mv ‘efficient’ portfolio and stores them in the vector $w$; this represents the optimal percentage of the investors capital invested in each index. Thus, the inputs are the expected returns of the assets and the variance-covariance matrix. These inputs are not known ex-ante and have to be estimated and these estimates are subject to estimation error which was discussed in Section 2.3.1. Solving this optimisation problem results in a mean-variance efficient portfolio. We can construct the mean-variance efficient frontier by solving this optimisation problem for a range of target
returns, see Fig 3.1. All permittable portfolios over $w$ are part of the *feasible set*. However, only the *minimum variance* portfolios for each level of target return are *mean-variance efficient*. Thus, the *efficient frontier* is comprised of the set of *mean-variance efficient* portfolios with each portfolio on the frontier being an optimal trade-off between risk and expected return. Portfolios in the *feasible set* that fall below the frontier are inefficient and therefore of no interest to an investor with mean-variance preferences. It is not possible to obtain a portfolio that falls above the frontier.

![Figure 3.1: Efficient Frontier](image)

**3.3.1 Global Minimum-Variance Portfolio**

The global minimum-variance (gmv) portfolio is the leftmost point on the mv efficient frontier. It is the portfolio with the smallest variance achievable given the set of risky assets $N$. It represents a fully-invested portfolio with the lowest variance that can be achieved. The gmv portfolio has less
estimation error compared to all other portfolios on the efficient frontier as the portfolios weights are computed purely by the variances and covariances, hence omitting the largest potential cause of estimation error which is the $\mu$.

It is computed using the problem.

$$\min_w w^T \Sigma w$$

subject to the constraints

$$w^T l = 1, l^T = [1, 1, ..., 1]$$

which minimises the portfolio variance over a set of weights $w$. The weights for the gmv portfolio are calculated using the solution

$$w = \frac{1}{l^T \Sigma^{-1} l} \Sigma^{-1} l$$

Thus the mean expected return and volatility of the portfolio can be calculated using these weights [32].

### 3.3.2 Tangency Portfolio

The tangency portfolio is a fully-invested portfolio that maximises the sharpe ratio. It maximises mean return over volatility $\mu/\sigma$. It can be found by drawing a tangent line from the origin to the efficient frontier Fig 3.1. It is solved using the problem

$$\max_w \frac{\mu_P}{\sigma_P}$$

$$\min_w w^T \Sigma w$$

with solution
\[ w = \frac{1}{p\Sigma^{-1}p}. \] (3.23)

### 3.4 Bayes-Stein Estimators of Expected Return

As shown, the Markowitz framework proceeds as follows, firstly the moments (mean, variance) of a time series of historical returns (in our case 30 years of 10 index returns) were calculated and then the mean-variance optimisation problem is solved treating these estimates as true parameters. This certainty viewpoint of the parameters has been heavily criticized in the academic world (Brown 1976) [33]. The bs estimator was advocated (Jorian 1985) as it is a way of reducing estimation error associated with the expected returns and variance by shrinking the mean towards the global minimum variance portfolio Fig 5.2. Using a shrinkage factor, the bs procedure can be used to reduce the sensitivity of our results to these errors in our parameter estimates. This shrinkage factor is calculated by solving the equations:

\[ s = \frac{\lambda}{T + \lambda} \] (3.24)

\[ \lambda = \frac{(N + 2)(T - 1)}{(\mu - \mu_{gmv})\Sigma^T(\mu - \mu_{gmv})(T - N - 2)} \] (3.25)

where \( \mu \) is the mean return of the indices, \( \mu_{gmv} \) is the return of the gmv portfolio, \( T \) is the number of monthly returns and \( N \) is the number of equity indices [34].
Chapter 4

Data Analysis

4.1 Data

In analysing the performance of the mean-variance framework, ten global equity indices were chosen as displayed in Table 4.1. This historical data was sourced from Yahoo Finance [35] for the time frame February 1973 to December 2003. The chosen data timeframe purposely omitted the recent 2008 financial crisis and subsequent recession as during this timeframe all asset classes decreased in value and thus all correlations approached one thus diversifying was limited. The financial crisis was a 'black swan' event meaning it was impossible to predict and would cause our results to deviate from what we would normally expect in terms of index returns and volatility. Thus omitting that time frame would give a clearer picture of the performance of the mv framework over time.

The data itself consists of 370 monthly prices for each of the ten indices as well as the corresponding dates. The data was cleaned and sorted so any empty days were omitted and imported into an excel file. Of the ten indices
Table 4.1: Indices

<table>
<thead>
<tr>
<th>Germany</th>
<th>U.S.</th>
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<tbody>
<tr>
<td>Japan</td>
<td>France</td>
</tr>
<tr>
<td>Ireland</td>
<td>Italy</td>
</tr>
<tr>
<td>Netherlands</td>
<td>U.K.</td>
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<tr>
<td>Canada</td>
<td>Switzerland</td>
</tr>
</tbody>
</table>

the best performing index in terms of percentage annual return for the time frame was Ireland and the worst performing index was Japan. This performance is displayed in the growth of a dollar graph Fig 4.1. Thus an investor practicing a passive investment approach (buy and hold) investing a dollar in the Irish index in 1973 would have seen their investment grow to over 80 dollars by the end of 2003 versus just below 10 dollars for the same investment in the Japanese index. This figure showcases the rapid growth of the Irish economy versus the rest of the world in recent times due to the Celtic Tiger era. The huge effect the Dot Com collapse of 2000 had on the indices can also be made out as well as the 1987 crash 'Black Monday'. The graph shows the sharp decline in returns during these periods.

The reason behind choosing 10 indices for our portfolio in particular is that it is the number cited as being large enough for a portfolio to become free of unique/idiosyncratic risk - the risk of individual assets. According to Evans and Archer (1968) who concluded that relatively few stocks are needed to eliminate unique risk and achieve diversification [36]. Investors holding at least ten assets in their portfolio can utilise diversification benefits and mv optimisation to eliminate this unique firm specific risk.
Fig 4.1: Growth of a Dollar

Fig 4.2 is more interesting as it shows the annualised expected returns and corresponding volatility for each index. Although Ireland (IE) has the highest mean return it also had the second highest volatility, conversely Switzerland (CH) had one of the lowest mean returns and also one of the lowest volatility levels Fig 4.2. This is indicative to the Markowitz framework and the risk/reward trade-off which states that in order to achieve higher returns an investor must also take on higher risk (volatility) in the form of deviating returns and choose what point on the efficient frontier Fig 3.1 satisfies their utility function and level of risk aversion.

Table 4.2 contains the estimated correlation co-efficients of returns for all ten indices. The most highly correlated indices are the U.S. and Canada with a co-efficient of 0.7451. The lowest correlated indices used in our data are Italy and Japan with a correlation of .3021. We would expect to see diversification benefits associated with this portfolio of indices as they are
not perfectly correlated in fact some have quite low correlations. It would have been optimal if there were lower correlations between the indices as this would give the opportunity to decrease the variance of returns further but in reality finding negatively correlated equity indices is hard as generally all indices are somewhat correlated as shown in Table 4.2.
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Chapter 5

Portfolio Optimisation

Simulations

5.1 Efficient Frontier

To construct the efficient frontier for our 10 index data set we first computed the annualised mean return of each of the ten indices for the range 1st of February 1973 to the 1st of December 2003. This was calculated by first getting the monthly log returns using prices of each index, then converting them to arithmetic returns and averaging them, this resulted in the average monthly percentage return and finally multiplying the returns by 12 gave the annualised mean return of each index. Next the standard deviation of the monthly arithmetic returns for each index were calculated and then annualised by multiplying by the square root of 12. The correlations between the indices were then calculated and thus the estimated variance-covariance matrix (\( \Sigma \)) was created.

Using this data the efficient frontier was computed by varying the target
portfolio return over the window [9.24%, 14.3%], as these percentages represented the return of the gmv portfolio and the max return of the highest performing index(Ireland) respectively. We specified 25 evenly spaced target returns over the window which resulted in 25 portfolios each portfolio representing the lowest variance of returns for that target return. The weights for each of 25 portfolios were calculated using the closed form solutions explained in chapter 3 equation 3.14. These weights were then substituted into the formulae for the expected return and standard deviation for each portfolio and thus the efficient frontier was plotted. With the x-axis representing the annualised mean return(%) and the y-axis representing the annualised standard deviation(%) or volatility of the returns.

![Figure 5.1: Efficient Frontier](image)

Each of the circles represents the annual mean return and volatility for each of the optimal 25 portfolios Fig 5.1. Each circle contains the lowest risk attainable for that level of target return. The red dot represents the gmv portfolio which as previously mentioned represents the lowest standard deviation of
returns that can be achieved for this data set. The green dot represents the
tangency portfolio. The tangency portfolio is highlighted as it is the portfolio
that maximises the ratio of excess return to standard deviation $\mu/\sigma$ (Sharpe
ratio) [37].

If we compare the return and volatility of one of the portfolios to an index
with a similar return we can see the benefit in terms of risk reduction. Take
for example Ireland(IE), over the dataset’s 30 year time frame the Irish index
had a computed expected annual mean return of 17.14% and a volatility of
23.37% Fig 4.2. If we compare that to an optimal portfolio that lies on the
efficient frontier for example portfolio number 25(rightmost portfolio) which
also has an annual expected return of 17.14%, but an annualised volatility of
just 18.46%. Table 5.1 shows the comparison of some of the indices against
optimal portfolios on the efficient frontier with the same expected return.
The results show the significant benefit in terms in terms of risk reduction
of using mv optimisation.

### 5.1.1 Efficient Frontier: Bayes-Stein estimator

As mentioned previously, one of the main criticisms of the Markowitz mv
framework is the estimation error associated with $\mu$ as our data is only sam-
ple data (albeit over 30 years) and therefore $\mu$ is a sample estimate. The mv framework treats these sample estimates as true values, this approach has been heavily criticised. To account for this parameter uncertainty, the sample mean is shrunk towards the gmv portfolio by an amount dependant on the shrinkage factor(%) thus reducing the estimation error in the portfolios as the gmv portfolio is computed using the variance-covariance matrix. Figure 5.2 shows the original efficient frontier in blue and the frontier which incorporates the bs estimator in red. The estimator is computed using the formulae in equation 3.24 and 3.25 and is substituted into Algorithm 2 to compute the weights for each of these Bayes-Stein portfolios. This algorithm describes the process for the calculation of these bs portfolios.

This new efficient frontier shrinks the portfolios towards the gmv portfolio and can be interpreted as more accurate as it is accounting for the estimation error in the parameter inputs.
Algorithm 2 Algorithm Bayes-Stein Efficient frontier Portfolios

\begin{algorithm}
\For{$i = 1$ to $numPorts$}{
    \{analytical solution for optimal portfolio weights\}
    \[w = g + h \cdot ER_{mv}(i)\]
    \{solution for standard deviation of each optimal portfolio \}
    \[STD_{mv}(1,i) = \sqrt{w' \cdot C \cdot w}\]
    \{solution for bs weights where $W_{mvg} = gmv$ portfolio weights\}
    \[w_{bs} = shrinkageFactor \cdot W_{mvg} + (1 - shrinkageFactor) \cdot w;\]
    \{expected return of each bs portfolio with bs shrinkage factor\}
    \[ER_{bs}(1,i) = shrinkFactor \cdot ER_{mvg} + (1 - shrinkFactor) \cdot retAvg \cdot w_{bs}\]
    \{calculating standard deviation of each optimal bs portfolio \}
    \[STD_{bs}(1,i) = \sqrt{w_{bs}' \cdot C \cdot w_{bs}};\]
}\end{algorithm}

Also shown are the weights for the gmv portfolio Fig 5.3 displaying the percentage of the investors capital invested in each index, which gives the lowest attainable variance for their target return. These weights are dominated by the Canadian(CA), Japanese(JP) and the USA(US) index as these are the indices with the lowest volatility. The negative weights discussed in Chapter 4 represent shorting of that specific index, indices with relatively high volatility are shorted e.g. France and Ireland. This is due to the fact that the gmv portfolio is not concerned with expected returns but only the volatility.

5.2 Out-of-Sample Portfolio Performance

A script coded in Matlab was used to iteratively backtest the performance of four of the significant portfolios over a moving window (first 150 months of data) to assess the out-of-sample performance of each of them based on
a similar method advocated by Kolusheva (2008) [38]. The out-of-sample period being the remaining 220 months of the 30 year time frame. The sample mean of these portfolios is the mean of the first 150 months of returns for the 30 year data set described as the moving window, with an estimation of the out-of-sample portfolio returns using this moving window over the entire 30 year data set. The four portfolios consisted of an Equally-Weighted, GMV, Tangency and Tangency with Bayes-Stein estimator portfolio. Each iteration of the backtest consisted of the following:

- Beginning with time $t=M$ (where $M=150$; length of moving window) the expected return, variance and covariance matrix inputs for each of the 4 portfolios are estimated over the preceeding $M$ months.

- Solve the optimisation problem for the 4 key portfolios.

- Calculate the return on each of the 4 portfolios in period $t+1$ based on the optimal weights computed from previous step. So for the first
iteration of the backtest this would be the return of the portfolio for time \( t=151 \) when \( M=150 \).

Moving the window then consisted of dropping the earliest monthly return and adding the return for the next month. Therefore after the first iteration the moving window is from \( t=2 \) to \( t=151 \). Thus the moving window size remains at a constant length of 150. The 3 steps previously described are then repeated resulting in the return for each of the portfolios for time \( t=152 \). This iterative backtest is then repeated until the end of the 30 year data set (i.e. \( T=370 \)) resulting in \( T-M \) (370-150) monthly out of sample returns for each of the 4 key portfolios.

![Figure 5.4: OOS Performance of Key Portfolios](image)

Fig 5.4 shows the results of this out-of-sample performance backtest of each portfolio in terms of the growth of a dollar. The results show that as expected the equally-weighted naive portfolio performed the best with 1 dollar invested in 1985 would be worth over 7 dollar’s in 2003 (18 year out-of-sample period).
Table 5.2:

<table>
<thead>
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<th></th>
<th>Equal Weighted</th>
<th>GMV</th>
<th>Tangency</th>
<th>Tangency-bs</th>
</tr>
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<tbody>
<tr>
<td>Annualised Mean Return</td>
<td>0.1210</td>
<td>0.1021</td>
<td>0.1105</td>
<td>0.1066</td>
</tr>
<tr>
<td>Annualised Volatility</td>
<td>0.1532</td>
<td>0.1466</td>
<td>0.1560</td>
<td>0.1509</td>
</tr>
<tr>
<td>Sharpe Ratio</td>
<td>0.7898</td>
<td>0.6964</td>
<td>0.7850</td>
<td>0.7065</td>
</tr>
</tbody>
</table>

The gmv portfolio performed worst as expected while the tangency portfolio outperformed the tangency with bs portfolio. Although the gmv portfolio performed the worst in terms of percentage return it achieved the lowest variance Table 5.2 compared with the other portfolio’s. This portfolio would appeal to many risk-averse investors who wish to minimise the volatility in their returns and also because it has less estimation error in the parameter inputs. The tangency portfolio also had significantly higher variance in its returns compared to the tangency-bs portfolio showing the benefit in terms of risk reduction by using the bs estimator. This is expected as the tangency-bs portfolio has been shrunk towards the gmv portfolio which has the lowest variance and thus reducing the variance of the tangency-bs portfolio.

The shrinkage factor varied over time during the out-of-sample backtest for the tangency-bs portfolio as the expected returns in the moving window changed thus affecting the shrinkage factor. The changing of the shrinkage factor is displayed in graph Fig 5.5. As you can see the emphasis put on this factor varied between 0.2% and 0.5%. The same approach for the backtest was then applied to each of the 25 mv efficient portfolios to assess their out-of-sample performance over the moving window. The results are displayed in Fig 5.6.
Figure 5.5: Shrinkage Factor

Figure 5.6: OOS Performance of MV Portfolios
Chapter 6

Coding

An integral part of my project was the development of a program to independently handle the users desired data set of asset prices and corresponding dates and to carry out an analysis, and provide key metrics and information relative to a potential investor. The program objectives are briefly discussed first, followed by an outline of the program design and then by the actual Matlab [39] implementation, discussing the relative API’s aswell as the event driven GUI and it’s main features.

6.1 Program Objectives

The objective of this Matlab application is to assess the performance of the Markowitz mean variance framework as well as the effect of a Bayes-Stein estimator for a data set specified by the user. Thus allowing the investor to select a particular portfolio that matches their risk/return appetite and to obtain the weights necessary to attain this risk/return. The objective is also to display key metrics and performance graphs of use to a potential investor and display them in the form of a graphical user interface, allowing the user
to select different portfolios and observe the results. In addition the investor can view the out-of-sample performance of their selected portfolio as well as the following key portfolios over a moving window.

- Global Minimum Variance Portfolio
- Equally Weighted Portfolio
- Tangency Portfolio
- Tangency with Bayes-Stein Portfolio

The application should be simple to use and require no programming knowledge to run and interact with. The user should be able to navigate easily between tabs and press buttons without unexpected errors. The objective is to display relevant data and graphs that can be of use to a real life portfolio investor, and that the application can be used for any data set of monthly prices in excel form.

6.2 Program Design

The application was developed to carry out a statistical and performance analysis on a chosen dataset. The application was implemented in the Matlab language. The reason for choosing Matlab was that it supports object orientated programming which makes it far easier to create complex applications such as this one and greatly reduces the need of replicating code. Matlab also offers a multitude of financial API’s for basic statistical calculations such as the variance-covariance matrix. Although I had no experience with Matlab what made it very appealing is the fact it is the foremost software used by financial and algorithmic traders. The API’s offered numerous
efficient functions and classes for many of the more basic operations performed in the application.

The first part of the application’s implementation is the excel data extraction. The application was developed to reduce the need to process the data before being extracted, the only formatting needed is for the dates to be in column one of the excel file and the index prices in the subsequent columns—see Fig 6.3. The data used in this project was downloaded from the Yahoo finance website which is freely available and loaded into an excel file. The user has the option to either run a Matlab script that focuses on the specific chosen dataset discussed in this paper. The script will analyse the data set and perform multiple statistical and performance analyses, ultimately outputting twelve graphs displaying all relevant financial information to an investor. This script allows no user interaction or the ability to select and view characteristics of specific portfolios. The overview of the process from the perspective of the user is displayed in Fig 6.1.

Figure 6.1: Script Overview

The user also can run a Matlab Application, where again the user reads in
a data file along with other variables and a multi-tab event driven GUI is displayed. This application allows the user to interact with the data and extract the necessary information through mouse clicks. There are a multitude of buttons/sliders which when clicked display key information to the user. One such function is the ability to select a particular portfolio and view the information and performance of that portfolio. The process is displayed in Fig 6.2. The most important graphs displayed from this program are in the 'Out-of-Sample’ tab when the 'OOS Performance’ button is pushed. This button outputs the out-of-sample performance of the four key portfolios shown and discussed in Chapter 5 and displayed in Fig 5.4. The selection of a portfolio populates the 'Key Metrics' tab with information such as the annualised mean return and volatility as well as the weights for the selected portfolio. All of which are of great importance to an investor.

![GUI Overview Diagram](image)

**Figure 6.2: GUI Overview**
6.3 Matlab API’S

Using the Matlab API’s provided many efficiencies in the development of the program and removed the need for writing some complex formulae. Matlab API’s used

- Matlab’s Financial Toolbox- The toolbox provides a comprehensive suite of portfolio optimisation and analysis tools for performing capital allocation, asset allocation, and risk assessment [40].

- Matlab Publishing- Provides the ability to publish your output in html file format, as well as the ability to create readable external documentation of your code and publish your code to Latex with LaTeX markup [41].

6.4 Data Import and Handelling

The user specifies the excel data file to be extracted as well as other variables specified in Figure 6.4. The excel data should format the data similar to Fig 6.3 with the dates located in the first column and the index prices in the subsequent columns. Next using the ’xls read’ function built into Matlab, the program loads the file and the variables(sheet number, column and row etc.) into memory in the ’GUI’ class. The sheet number defines what sheet of the excel file is to be extracted and the row and column define the data region in the sheet to be extracted. In Fig 6.3 the data region would be A4 to K50 as we are only interested in the dates and index prices.

The program then creates an object of the class ’myportfolio’ and using a constructor initialises the object with those variables allowing the object to
call methods of that class. Once the data is loaded into memory it is split up into the dates and index prices and the arithmetic returns of each of the indices are then calculated. Methods are then called which calculate the inputs for the efficient frontier as well as the bayes-stein frontier.

6.5 GUI

The GUI itself consists of 5 tabs see Appendix 2. After the data is imported and handled as described above the GUI opens and displays the first tab.

---

Figure 6.3: .xls File
named ‘Overview’ where a basic overview of the function of each tab is given. Each subsequent tab consists of the following basic overview:

- **Overview Tab**: Brief description of the application and the tabs functionality.

- **Data Tab**: Displays the imported data file so the user can easily view the prices and dates of each imported index price series.

- **Portfolio Optimisation Tab**: Contains a crosshair to select a portfolio on the frontier, when a portfolio is selected graphs are displayed relating to the weights of the portfolio and the out-of-sample portfo-
lio performance relative to a benchmark. Includes a slider button to change the shrinkage factor and resulted impact on the efficient frontier is displayed.

- Information Tab- Key Metrics related to the the selected portfolio are displayed such as the annualised return and volatility as well as the weights for each index. The correlation table of the data set is also displayed. A table is generated showing the out-of-sample performance of the 4 key portfolios.

- Out of Sample Tab- This tab contains 4 buttons which graph the growth of a dollar for each index, efficient frontier, bs efficient frontier and the out-of-sample performance of the 4 key portfolios. There is also a button that generates a html report of the selected portfolio.

### 6.6 Critical Analysis

Figure 6.5 contains a brief SWOT analysis of the application.

<table>
<thead>
<tr>
<th>Strengths</th>
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<td>Efficiency</td>
<td>GUI display packages</td>
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<tr>
<td>Financial API’s and Toolboxes</td>
<td>No previous experience with Matlab</td>
</tr>
<tr>
<td>Supports OOP</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Opportunites</th>
<th>Threats</th>
</tr>
</thead>
<tbody>
<tr>
<td>Java calling</td>
<td>Private Industry Applications</td>
</tr>
<tr>
<td>Extending Application</td>
<td></td>
</tr>
</tbody>
</table>
The application would benefit from a sleeker display as the packages that come with Matlab are not comparable to that of other languages such as Java. The application is efficient as it supports OOP and thus no need for replication of code. An object calls methods only when the user interacts with the GUI and the results from those method calls are displayed. The object created stores those results so as they can be accessed at any time quickly and efficiently. The application would also benefit with improved robustness and perhaps an easier way of importing data and not having to specify so much variables such as the rows and columns. Other file extensions would also be useful and not limiting them to just .xls inputs. Importing the data straight from Yahoo Finance would also be an excellent add on to the application.
Chapter 7

Conclusions and Future Work

7.1 Findings

This paper gives a detailed discussion and analysis of the mean-variance framework over a 30 year period (1973-2003) and seeks to answer the question of: How should an investor optimally allocate their capital?.

The paper shows the benefits of diversifying and allocating capital across a portfolio of indices rather than just investing all their capital in a single index. The paper also presents the standard approach for allocating across a portfolio namely the Markowitz mean-variance framework. Showing the efficient frontier for the data set as well as the incorporation of a bs estimator into the mean-variance optimisation process thus showing the improved performance in terms of risk reduction (decreased variance of returns) when an investor allocates their capital based on the estimator.

The out-of-sample performance of the mean-variance process is also tested as well as the out-of-sample performance of the equal weighted, global minimum variance, tangency and tangency bayes-stein portfolios. As expected
the equal weighted portfolio outperforms the other portfolios which supports the estimation error associated with the mv portfolios. Confirming the evidence that mean-variance portfolios perform poorly out of sample. This is due to the extreme weights these portfolios have displayed in Fig 7.1. as they are extremely sensitive to the parameter inputs especially the estimation of expected returns.

The out-of-sample performance of the gmv portfolio is superior in terms of risk reduction compared to all other portfolios including the equal weighted. Therefore the gmv portfolio is proved to be more reliable and having less estimation error and backs up the argument of using a bs estimator by shrinking the sample mean towards the gmv portfolio.

Paper also details the Matlab application developed as well as its functional

![Figure 7.1: Index Weights for the 25 OOS portfolios](image)

uses. A basic overview of the application is given as well as a critique of the
an application.

7.2 Extensions

An interesting extension to this project would be accounting for the non-normal distribution of prices discussed as a criticism of the mean-variance framework in Chapter 2 as well as the underestimation of extreme outliers that deviate far from the mean. One distribution to account for this would be modelling asset returns using a multivariate student T distribution as advocated by Shaw(2011) [43].

Going beyond the first two moments (mean, variance) and accounting for higher moments such as skewness and kurtosis (fat tails) can lead to substantially different asset allocations than the Markowitz mv framework. By using the mean-conditional value at risk optimisation (MCVAR) the impact of fat tails and skewness (i.e. non normal distribution) can be accounted for and hence reduce the estimation error as shown by Xiong(2010) [44].

The ultimate aim is to extend this paper and application and implement these suggested extensions while pursuing an M.Sc in Finance next year in Trinity College Dublin bringing together my Computer Science and Finance background.
BIBLIOGRAPHY:


(16), (22) Black, F., Litterman, R. (1992), Global Portfolio Optimization. Financial Analysts Journal 48, 5; pg.28


(38) Kolusheva, D.,(2008), Out-of-Sample Performance of Asset Allocation Strategies


Appendix:

Appendix 1: Short Selling Definition- 'The selling of a asset that the seller does not own, or any sale that is completed by the delivery of an asset borrowed by the seller. Short sellers assume that they will be able to buy the stock at a lower amount than the price at which they sold short' [42].

Appendix 2: GUI TABS-

![GUI Tab](image-url)

Figure 2: GUI Tab
Figure 3: GUI Tab
### Figure 4: GUI Tab

#### Key Metrics and Information

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Figure 5: GUI Tab