Bayesian Discrete Latent Spatial Modelling of Crack Initiation in Orthopaedic Hip Replacement Bone Cement

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Abstract
In this paper we propose a spatial model for the initiation of cracks in the bone cement of hip replacement specimens. The failure of hip replacements can be attributed mainly to damage accumulation, consisting of crack initiation and growth, occurring in the cement mantle that interlocks the hip prosthesis and the femur bone. Since crack initiation is an important factor in determining the lifetime of a replacement, the understanding of the reasons for crack initiation is vital in attempting to prolong the life of the hip replacement.

The data consist of crack location coordinates from five laboratory experimental models, together with stress measurements. It is known that stress plays a major role in the initiation of cracks and it is also known that other unmeasurable factors such as air bubbles (pores) in the cement mantle are also influential. We propose an identity-link spatial Poisson regression model for the counts of cracks in discrete regions of the cement, incorporating both the measured (stress), and through a latent process, any unmeasured factors (possibly pores) that may be influential. All analysis is carried out in a Bayesian framework, allowing for the inclusion of prior information obtained from engineers and parameter estimation for the model is done via Markov chain Monte Carlo techniques.

Keywords: Orthopaedic hip replacement, crack initiation, identity-link Poisson regression, latent spatial process, Bayesian analysis, Markov chain Monte Carlo, Zero-inflated Poisson.

1 Introduction
Orthopaedic joint replacements are used to replace human joints that no longer function as they should. In particular, an orthopaedic hip replacement is used to replace the ball and socket components of the hip joint. The hip replacement operation is considered to be a successful operation, with studies reporting over 90% of replacements still functioning well after ten years (Huiskes and Verdonschot (1997)). Since the operation is a very common procedure, for many patients it is a failure and revision operations are not nearly as successful as primary ones (Malchau et al. (2000)). Also, the procedure is being carried out on younger patients, leading to the need for longer lasting replacements.

Typically the hip replacement consists of a prosthesis, usually metallic, being inserted into the medullary cavity of the femur bone which has been hollowed out. The metal prosthesis is a replacement for the “ball” portion of the joint. The “socket” portion of the joint is replaced with an artificial cup. The prosthesis is held in place by an acrylic polymer cement mantle that interlocks the prosthesis and the bone. This acrylic polymer cement is usually referred to as bone cement. See Figure 1(a).

The main reasons for failure of the hip replacement are infection and the mechanical loosening of the components. Due to improvements in surgical conditions, infection has been almost eliminated (cumulative revision rate for deep infection after ten years is 0.3%, Malchau et al. (2000)), leaving mechanical loosening of the components as the dominant mode of failure, in particular the mechanical loosening of the femoral prosthesis (Malchau et al. (2000)). Damage accumulation, consisting of the initiation and growth of cracks in the bone cement, is thought to be the main reason for the mechanical loosening, in particular in the femoral component of the implant, as it is here at the interface between the prosthesis and the cement that the greatest stress is experienced (Lennon (2002)). When in use, cyclic loads of several times a person’s bodyweight are experienced at the hip (Bergmann et al. (1993)) and relative to
the stresses that are applied at the hip, bone cement is a weak material (Huiskes (1993)). The cyclic loading results in compressive and tensile stresses along the femoral stem of the implant. Thus stress is a very important factor in determining whether a replacement will fail or not.

Maintaining the cement mantle is not just a matter of reducing the peak stress in the mantle though. Another factor to consider is the distribution of pores which form during the preparation of the bone cement when air bubbles become trapped in the cement. Lennon and Prendergast (2001) highlight the situation where the influence of cement porosity may dominate the effect of stress to such an extent that failure may occur, not at the location of the peak stress but instead where the pores are largest. In such cases, recording the peak stress would not give sufficient information for investigating crack initiation. Further evidence of the importance of porosity in causing failure of the replacement is given by retrieval studies which show a link between the porosity in the bone cement in the hip replacement and damage accumulation (Jasty et al. (1991); Culleton et al. (1993)).

The amount of porosity varies depending on the way in which the cement is mixed. Improvements in the mixing of the cement such as mixing under a vacuum and centrifuging, as opposed to manual mixing with a bowl and spoon, decrease the amount of air bubbles trapped but do not eliminate all pores (Wang et al. (1996)). According to Lennon (2002) the pores that do remain can often be very large (which can lead to early failure of the implant) in the case with vacuum mixing, or the pores may be heterogeneously distributed in the case of centrifuging.

Mathematical modelling provides an excellent tool for describing the process of damage accumulation, and various models have been proposed with previous approaches to modelling damage accumulation in the bone cement focusing on investigating lifetimes and reliability of replacements. For example, the fitting of a Weibull distribution in a survival analysis is carried out in Malchau and Herberts (1998). In Wilson (2005), a hierarchical model for crack initiation and growth is proposed and a Bayesian analysis is carried out. The choice of hierarchical model is based on the fact that the variability between specimens and within specimens can be attributed to different influential physical factors. An example of within-specimen variability given by Wilson (2005) is that of random variations in the distribution of pores in the cement. Between-specimen variability could be due to the random differences in the mixing of the cement, for example. A Poisson process, modelled as a function of local material properties of the cement, is proposed for the initiation of the cracks. This model is similar to that proposed by McCormack et al. (1998), where the same data were analysed. The type of models fitted are necessarily dependent on the type of data available. Unlike the data previously analysed by Wilson (2005) and McCormack et al. (1998) for example, the data presented for analysis here contain spatial information regarding the location of the observed cracks and spatially varying stress measurements, allowing for a very different type of analysis to be carried out. Unfortunately, accurate recording of pore locations and sizes has not been possible in the experiment from which our data come. Our aim in this paper is to model crack initiation by exploiting the spatial information contained in the data and also to incorporate in our model, through a latent (hidden) process, the influence that porosity may have on causing cracks to form.

We propose modelling the initiation of cracks in the bone cement with a Poisson identity-link regression model, incorporating both the observed stress measurements and a latent process that models crack initiation due to unmeasured factors, such as the pores previously mentioned. The model also incorporates a Gaussian kernel and through this kernel we propose estimating the range over which such latent factors would have an influence on crack initiation. All analysis is carried out in a Bayesian framework allowing for the inclusion of prior expert knowledge obtained from engineers together with the likelihood distribution for our model. Parameters of the model are estimated using Markov chain Monte Carlo techniques.

In the next section we describe the experimental model and the resulting data that we analyse. In Section 3 we introduce the model and Section 4 deals with inference of the model parameters. The results of both a simulation study and the application of our model to the available data are presented in Section 5 and we carry out model validation in Section 6. We conclude in Section 7 with a discussion of our model and the results obtained.

2 The Experimental Model and the Data

An experiment was carried out in a laboratory setting to mimic, with an experimental model, the behaviour of a hip replacement. The design of the experimental model ensured that it would behave in a
similar fashion to a real hip replacement in the human body but yet allow measurements of damage accumulation to be collected, i.e., the locations of cracks could be identified. Full details of the experiment can be found in Lennon (2002).

The model consists of a femoral stem encased between layers of cement and strips of cancellous bone, see Figure 1(b). These are then held in two aluminium side plates which offer support similar to that which would be given by the cortical bone in the human body. The side plates contain windows (one medial, one lateral) in which the cement is exposed and therefore available for observation. The bone cement that was used is a translucent cement, enabling cracks to be stained and viewed by light transmission.

In order to recreate what happens in the human body a stress loading was applied to the physical model. The cement layers were treated with a dye penetrant, allowing a magnified image of the cement surface to be projected onto a screen. Crack locations in the cement were identified and traced onto acetate transparencies. The transparencies were digitally scanned and image analysis was carried out in order to obtain the exact position of each crack, resulting in x, y coordinates for each crack location. The experiment was repeated for five identical specimens. Figures 3 and 4 (left-hand-side diagrams) show the cracks that initiated during the stress loading. It is important to note that although all specimens were prepared in an identical manner and subjected to the same treatment (as was physically possible), huge variability between specimens exits both in the spatial distribution of the crack locations and in the number of cracks that formed.

As well as individual crack locations, the data provided also include a finite element analysis of stress measurements (both compression and tension) in a typical specimen (only one set of stress measurements for all five specimens).

### 3 Modelling

An approach when considering data consisting of actual spatial coordinates, such as we have, is to aggregate the data over a grid/partition and then model the resulting count data (for example, Ickstadt and Wolpert (1997)). The grid, often referred to as a “lattice” (Cressie (1993)), denotes a countable collection of (spatial) sites and it is often possible to specify neighbourhood information for the lattice. In disease mapping, where the number of disease cases in each geographical region is of interest, the data are often supplied as count data (Elliott (2000)). A natural choice when dealing with count data is to use the Poisson distribution. A type of model sometimes used is the log-linear Poisson model. In this case the count data are modelled as Poisson and the logarithms of the intensity can then be modelled with a Gaussian random field. Unfortunately if the level of aggregation is not the desired level, the model does not scale so easily. An alternative is the identity-link Poisson regression model (Dobson (2002)) which is consistent under aggregation and refinement of the chosen partitions. Poisson regression models with identity-link functions have been used in various applications, for example, in spatial epidemiology (Best et al. (2000b)) and in the examination of forest inhomogeneity (Ickstadt and Wolpert (1997)).

#### 3.1 Poisson Model

We choose an arbitrary partition of both the lateral and medial windows by dividing each window into a grid of 22 polygons. See Figures 3 and 4 (right-hand-side diagrams). Our aim is to relate the count of cracks in each polygon to both the measured and unmeasured factors that are influential in crack initiation.

Let \( P_{ij} \) denote polygon \( j \) of specimen \( i, \ i = 1, \ldots, 5; \ j = 1, \ldots, 44 \). We denote by \( N(P_{ij}) = N_{ij} \) the count of cracks in polygon \( P_{ij} \). A natural choice when dealing with count data is to consider some form of Poisson model. We model the crack count in each polygon as having an independent Poisson distribution with some unknown intensity, \( \mu_{ij} \). Thus we have

\[
E(N_{ij}) = \mu_{ij} = A_j \lambda_{ij},
\]

where \( A_j \) is the area of polygon \( P_{ij} \) (note, not all \( A_j \) are equal) and \( \lambda_{ij} \) is the unit-area intensity. The influence that any explanatory variables have on the crack count \( N_{ij} \) is modelled by \( \mu_{ij} \) through \( \lambda_{ij} \).

Through kriging (Cressie (1993) and Diggle and Ribeiro (2000)) we can obtain the stress measurements at the centroids of each of the polygons. This allows us to relate the count of cracks in a polygon to a
single stress measurement at the centroid of the polygon. See Figure 2. We denote by $C_j$ and $T_j$ the compression and tension respectively at the centroid of polygon $P_{ij}$ for all $i = 1, \ldots, 5$. Note that only one of $C_j$ and $T_j$ will be non-zero for a given polygon centroid, as compression and tension cannot both be present at a given location. Both compression and tension have an effect on crack initiation and we would like to estimate each of their effects separately.

Another factor that has an impact on the initiation of cracks is the distribution of pores (air bubbles) in the bone cement. As detailed in Section 1 the mixing of the cement is a very important step during the operation as it is possible for air bubbles to become trapped in the cement while it is being prepared which in turn can have an impact on the initiation of cracks in the cement. As previously stated, in the experiment from which our data come, it was not possible to locate pores within the cement. Thus we do not have any idea as to their distribution in each of the specimens. We would like to incorporate this important factor in our model and to do this we model the unobserved distribution of pores using hidden or latent spatial variables, which will also model excess variability due to any other unobserved factors that may have been present in the specimens.

We model the dependence of the random unit area intensity $\lambda_{ij}$ on the explanatory variables (stress, latent) using a Poisson regression model with identity link. The identity-link is chosen as it is consistent under both aggregation and refinement of the grid. If, for example, a logarithmic-link was chosen, attempting to aggregate the regions of the grid would lead to products for the Poisson means of the new aggregated regions instead of sums, as would be desired. The unit-area intensity now has the following form:

$$\lambda_{ij} = \beta_1 C_j + \beta_2 T_j + \sum_{k=1}^{44} \omega_{jk} \gamma_{ik}, \quad i = 1, \ldots, 5; \quad j = 1, \ldots, 44,$$

where $\beta_1$ is the coefficient of compression and $\beta_2$ is the coefficient of tension. We choose to model the effect of compression and tension without a specimen effect. The reason being we have only one set of stress measurements for all of the specimens and so we do not expect to be able to model a specimen effect. $\{\gamma_{ij}\}$ is defined to be the set of latent factors with one factor for each polygon of each specimen. These latent factors represent the effect in polygon $P_{ij}$ of the unobserved spatially distributed factors that have an influence on crack initiation. The amount of influence that these factors have on causing cracks to form in a given polygon is governed by $\omega_{jk}$, a Gaussian kernel, i.e.,

$$\omega_{jk} = \frac{1}{2\pi \rho^2} \exp \left( -\frac{|d_{jk}|^2}{2\rho^2} \right),$$

where $d_{jk}$ is the Euclidean distance from the centroid of polygon $P_{ij}$ to the centroid of polygon $P_{jk}$. If a polygon is far away from the polygon whose crack count we are modelling then the kernel will be relatively small and so the influence from the latent factor in this polygon will be small. This influence is controlled by the parameter $\rho$. We make the following assumption: when considering the crack count in a particular polygon, the latent factors associated with the polygons in the other window of this specimen do not have any influence in causing cracks to form in this polygon. We make this assumption as there is no physical link between the windows in the experimental model, see Figure 1(b). Thus, in Equation 2 the kernel $\omega_{jk} = 0$ if polygon $P_{ik}$ is not in the same window as polygon $P_{ij}$. Also, $\rho$ is not chosen to be specimen specific. The range of influence of the latent factors would not be expected to vary between specimens, even though the distribution of the latent factors may vary greatly between specimens.

4 Inference

4.1 Inference for $\beta_1, \beta_2, \{\gamma_{ij}\}$

The parameters $\beta_1, \beta_2, \{\gamma_{ij}\}$, and $\rho$ are all uncertain. $\beta_1$ and $\beta_2$ quantify the influence of compression and tension respectively, and the $\{\gamma_{ij}\}$ quantify the influence of unobserved, spatially varying factors on crack formation. The parameter $\rho$ indicates over what distance the latent spatial variables have an influence. All of these parameters are of interest and we would like to estimate each of them. We will carry out all inferences in a Bayesian framework, (see Lee (1997) for an introduction to Bayesian statistics).
The joint likelihood together with independent prior distributions for each of the unknown parameters $(\pi(\beta_1), \pi(\beta_2), \pi(\gamma_{ij})), \pi(\rho))$, gives, up to a constant of proportionality, the following joint posterior distribution:

$$
P(\beta_1, \beta_2, \gamma_{ij}, \rho|\{N_{ij}\}) \propto \prod_{ij} \left\{ \frac{\exp(-\mu_{ij})}{N_{ij}!} \pi(\gamma_{ij}) \right\} \pi(\beta_1) \pi(\beta_2) \rho(\rho).$$

(3)

For the coefficients of compression and tension we choose independent prior distributions as physically compression and tension will not both be present in a given polygon. For each of $\beta_1, \beta_2, \gamma_{ik}$ we choose Gamma priors. One reason for choosing Gamma priors is that the mean of the Poisson must be non-negative; another is our belief that each of these corresponding factors positively influences the formation of cracks. We also have reason to believe from communications with engineers that tension stresses have a greater impact on crack initiation than do compression stresses and our choice of prior distributions should reflect this knowledge. We choose the following priors:

$$
\pi(\beta_1) \sim \text{Gamma}(\alpha_1 = 1, b_1 = 0.1),
$$

$$
\pi(\beta_2) \sim \text{Gamma}(\alpha_2 = 3, b_2 = 0.1),
$$

$$
\pi(\gamma_{ij}) \sim \text{Gamma}(\alpha_g = 1, b_g = 0.1), \forall i, j.
$$

All terms in Equation 3, except for $\pi(\rho)$, have now been defined. We will return to the choice of distribution for $\pi(\rho)$ in the next section.

We use the technique of data augmentation (Tanner and Wong (1987)) to obtain the full conditional distributions for $\beta_1, \beta_2$ and $\gamma_{ij}$ in known form. Consider $N_{ij}$ the count of cracks in polygon $j$ of specimen $i$, $N_{ij} \sim \text{Poisson}(\mu_{ij})$ as given by Equations 1 and 2. We introduce a new set of random variables $\{N_{ijk}\}$, $l = 1, \ldots, 46$, by breaking up the count $N_{ij}$ into the number of cracks that are attributable to compression, tension and the latent factors, $N_{ij1}, N_{ij2}, N_{ij3}$, and $N_{ijl}, l = 3 \ldots 46$, respectively, with $\sum_{l=1}^{46} N_{ijk} = N_{ij}$. These new random variables are obtained as follows:

$$
N_{ij1} \sim \text{Binomial}\left(N_{ij}, \frac{\beta_1 C_j}{\lambda_j}\right),
$$

(4)

$$
N_{ij2} \sim \text{Binomial}\left(N_{ij} - N_{ij1}, \frac{\beta_2 T_j}{\lambda_j - \beta_1 C_j}\right),
$$

(5)

$$
\vdots
$$

$$
N_{ij45} \sim \text{Binomial}\left(N_{ij} - N_{ij1} - \cdots - N_{ij44}, \frac{\gamma_{ijk} \omega_{43} \omega_{43}}{\gamma_{ijk} \omega_{43} \omega_{44} + \gamma_{ijk} \omega_{44} \omega_{44}}\right),
$$

(6)

and

$$
N_{ij46} = N_{ij} - \sum_{k=1}^{45} N_{ijk}.
$$

(7)

It can be shown that $N_{ij1} \sim \text{Poisson}(\beta_1 C_j A_j)$, $N_{ij2} \sim \text{Poisson}(\beta_2 T_j A_j)$, and $N_{ijk} \sim \text{Poisson}(\gamma_{ijk} \omega_{jk} \omega_{jk} A_j)$, $k = 3, \ldots, 46$. The joint likelihood for the augmented data now takes the following form:

$$
L(N_{ij1}, N_{ij2}, \{N_{ijk+2}\}|\beta_1, \beta_2, \{\gamma_{ij}\}, \rho) = \frac{\exp(-\beta_1 C_j A_j)(\beta_1 C_j A_j)^{N_{ij1}}}{N_{ij1}!} \times \frac{\exp(-\beta_2 T_j A_j)(\beta_2 T_j A_j)^{N_{ij2}}}{N_{ij2}!} \times \prod_k \left\{ \frac{\exp(-\gamma_{ijk} \omega_{jk} \omega_{jk} A_j)(\gamma_{ijk} \omega_{jk} \omega_{jk} A_j)^{N_{ijk+2}}}{N_{ijk+2}!} \right\},
$$

$$
k = 1, \ldots, 44.
$$

This leads to a new form for the posterior distribution and on examining the full conditional distributions for each of the parameters $\beta_1, \beta_2$, and $\gamma_{ij}$ it can be seen that each is Gamma distributed, (see Appendix
In order to investigate what is the best approach for our model we explore an initial examination of the parameters, (Best et al. (2000b), Wolpert and Ickstadt (1998a)). For this parameter in a Bayesian context, choosing a Lognormal prior with appropriate mean and variance where \( \rho \) treated the parameter \( \mathbf{P} \).

The use of this type of kernel in an identity link spatial regression model has been considered by other authors (Best et al. (2000a), Best et al. (2000b), Wolpert and Ickstadt (1998a)). Previous models have treated the parameter \( \rho \) as fixed or certain (Best et al. (2000a)), or have considered carrying out inference for this parameter in a Bayesian context, choosing a Lognormal prior with appropriate mean and variance parameters, (Best et al. (2000b), Wolpert and Ickstadt (1998a)).

In order to investigate what is the best approach for our model we explore an initial examination of the parameter \( \rho \). We consider the log-likelihood function for specimen \( i \):

\[
\mathbb{P}(\{N_{ij}\}|\beta_1, \beta_2, \{\gamma_{ij}\}, \rho_m) = \sum_j \{-\mu_{ij} + N_{ij} \log(\mu_{ij}) - \log(N_{ij})\},
\]

where

\[
\mu_{ij} = (\beta_1 C_j + \beta_2 T_j + \sum_k \omega_{jk} \gamma_{ik}) A_j,
\]

\[
\omega_{jk} = \frac{1}{2\pi \rho_m^2} \exp \left\{ -\frac{|d_{jk}|^2}{2\rho_m^2} \right\},
\]

for the range of \( \rho \) values \( 0.01 = \rho_1 < \rho_2 < \ldots < \rho_m < \ldots < \rho_M = 20 \). This was done as follows. For each \( \rho_m \) we calculated

\[
\hat{\mathbb{P}}(\{N_{ij}\}|\beta_1, \beta_2, \{\gamma_{ij}\}, \rho_m) = \frac{1}{R} \sum_{r=1}^R \mathbb{P}(\{N_{ij}\}|\beta_1^{(r)}, \beta_2^{(r)}, \{\gamma_{ij}\}^{(r)})],
\]

where \( \beta_1^{(r)}, \beta_2^{(r)}, \) and \( \{\gamma_{ij}\}^{(r)} \) are the parameter estimates obtained at each iteration \( r \) of the MCMC algorithm using Gibbs samplers, when \( \rho \) is fixed at the value \( \rho_m \). Thus \( \hat{\mathbb{P}}(\{N_{ij}\}|\beta_1, \beta_2, \{\gamma_{ij}\}, \rho_m) \) is an estimate of \( \mathbb{E}[\mathbb{P}(\{N_{ij}\}|\beta_1, \beta_2, \{\gamma_{ij}\})] \) with respect to the posterior distribution \( \mathbb{P}(\beta_1, \beta_2, \{\gamma_{ij}\}|\rho_m, \{N_{ij}\}) \). For each \( \rho_m \) we then plot \( (\rho_m, \hat{\mathbb{P}}(\{N_{ij}\}|\beta_1, \beta_2, \{\gamma_{ij}\})) \) as can be seen in Figure 5. The log-likelihood was calculated by running the MCMC algorithm, Section 4.3, for each of the specimens individually, with \( \rho \) having a fixed value, for an appropriate burn-in period. One thousand samples were then taken for each of \( \beta_1, \beta_2, \) and \( \{\gamma_{ij}\} \) and used in the estimate (Equation 8). Figure 5 clearly shows that, for each of the five specimens, values less than approximately 2 are to be favoured for \( \rho \). As \( \rho \) becomes larger, i.e., the kernel becomes more diffuse, the log-likelihood decreases. This is as we would expect in light of engineering intuition that the influence of the latent factors is of a localised nature. Based on the information from the likelihood and from the range over which it is believed the latent variables have an influence, it would be possible to treat \( \rho \) as fixed at some appropriate value and carry out inference for the other parameters.

Instead, we choose to carry out inference for the parameter \( \rho \), rather than fixing this parameter at some particular value. This allows for more flexibility in our model. We include a Lognormal prior for \( \rho \) and Figure 6 shows various Lognormals parameterised by their means and variances. Choosing one of these Lognormals as a prior distribution has the effect of making large values for \( \rho \) less likely. The prior parameters should reflect the beliefs as to the range over which interactions from latent factors are to be expected. For a given value of \( \rho \), most of the Gaussian kernel will lie within a radius \( 2\rho \) of the parameter.
centroid of the polygon. A polygon is approximately 6 units across. We would like a prior for \( \rho \) to allow influence outside a given polygon and so the prior mean for \( \rho \) should reflect this. We choose the following Lognormal prior for \( \rho \), \( \pi(\rho) \sim \text{Lognormal}(1, 1.4) \), giving a mean of \( \approx 7.2 \) units, allowing for influence outside a given polygon. As was noted by Best et al. (2000a), the full conditional distribution for \( \rho \) is complex and not of known form, thus it does not facilitate the use of a Gibbs sampler. Instead we use a Gaussian random walk Metropolis update (see Appendix A).

4.3 MCMC Algorithm

The general structure of the MCMC algorithm takes the following form

1. Initialise \( \beta_1, \beta_2, \{\gamma_{ij}\} \), and \( \rho \). This can be done by choosing values from the prior distributions for each of these parameters.
2. Calculate the Gaussian kernel \( \omega_{ij} \) for each pair of polygons \( P_{ij} \) and \( P_{ik} \).
3. Simulate \( N_{ijk} \), see Equations 4, 5, 6, and 7.
4. Update \( \beta_1, \beta_2 \) and \( \{\gamma_{ij}\} \) using Gibbs samplers.
5. Update \( \rho \) using a Gaussian random-walk Metropolis step.
6. Repeat steps (a)-(e), once convergence has been achieved samples of all parameters can be obtained.

5 Results

5.1 Simulation Study

In order to investigate the identifiability of the parameters, in particular \( \rho \), we carried out a simulation study. Using the statistical package R, we fixed the parameters at known values, together with fixed compression and tension stresses, and simulated count data from a Poisson distribution for 44 polygons of the same area as each of the polygons in the experimental model. Inputing the resulting simulated count data, we used the MCMC algorithm detailed above in order to obtain estimates of our known parameters. The data were simulated with the following values: \( \beta_1 = 0.01, \beta_2 = 0.03 \) and \( \rho = 3 \). See Figure 7 for image plots of the stresses, \( \gamma \)'s, simulated counts, and the resulting \( \gamma \) estimates. Table 1 presents median estimates of the parameters: \( \beta_1, \beta_2 \) and \( \rho \), together with 5%, 95% quantiles. All true parameter values are well within the range of the quantiles, indicating good estimation of the parameters by the algorithm. Comparing Figures 7b and 7d gives an indication as to how well the \( \gamma \)'s are estimated, again all estimates appear to be reasonable.

5.2 Results for the 5 Specimens

We also present in Table 1 the estimates of the \( \beta \) parameters and the parameter \( \rho \) based on the experimental data for the five specimens. The estimates for the parameters are based on sample values obtained from the posterior distribution by running the MCMC algorithm detailed in Section 4.3. We computed 15,000 iterations of the program attributing the first 3,000 to burn-in. We examined each of the chains visually in order to inspect for lack of convergence. The chains appeared to have converged. We also ran multiple independent chains from various starting points. No evidence for lack of convergence was found. We present the estimates for the latent parameters in the form of image plots. For each polygon we present the posterior median value of \( A_j \sum_k \omega_{jk} \gamma_{ik} \), obtained from the MCMC algorithm. These plots show an estimate of the latent contribution to the intensity, see Figure 8. In Figure 9 we present the median intensity for each of the polygons of each of the five specimens.
6 Model Validation

6.1 Posterior Predictive Distribution for Counts

In order to carry out model validation we would like to examine how well our model would do at predicting counts of cracks in each of the polygons. To do this we calculate the posterior predictive distribution for the count of cracks in each polygon using the Rao-Blackwellized estimator (Casella and Robert (1996); Gelfand and Smith (1990)):}

\[
\hat{P}(N_{ij}|\{N_{ij}\}) = \frac{1}{R} \sum_{r=1}^{R} P(N_{ij}|\{N_{ij}\}, \beta_1^{(r)}, \beta_2^{(r)}, \rho^{(r)}, \{\gamma_{ij}^{(r)}\}),
\]

where \(R\) is the total number of iterations after burn-in. This was carried out as follows

\[
P(N_{ij} = n|\ldots) = \frac{\exp\left(-\mu_{ij}\right)\left(\mu_{ij}\right)^n}{n!} = f_{rn},
\]

and the estimator is given by the following:

\[
\hat{P}(N_{ij}|\{N_{ij}\}) = \frac{1}{R} \sum_{r=1}^{R} f_{rn} = \bar{T}_n, \quad n = 0, 1, \ldots
\]

We then calculate the cumulative sum of \(\bar{T}_n\) and obtain 95% quantiles for the predicted counts. This was done for the count in each polygon of each specimen. For demonstration, in Figure 10 we show for each of the polygons of Specimen 3 the actual count and the median posterior predicted count (and 95% quantiles), based on this estimate. Most of the predicted counts are very close to the actual counts, being well within the quantile ranges. The polygons with zero counts are almost all over-estimated, suggesting the model is not such a good fit for zero counts, (see Section 6.3 where we address this issue).

6.2 Cross-Validation Predictive Density

We carry out a cross-validation analysis by omitting the count, for each polygon in turn, and examining the resulting predictive densities. It is not possible to omit a full specimen from the analysis as the latent factors, the \(\gamma_{ij}\)'s are specimen specific. If the model is a good fit for our data then the predictive densities should contain the omitted counts. For our purposes the set of cross-validation densities is given by \(\{P(N_{ij}|\{N_{-ij}\})\}\) where \(\{N_{-ij}\}\) denotes all counts except the count \(N_{ij}\) for polygon \(P_{ij}\). The density \(P(N_{ij}|\{N_{-ij}\})\) gives an indication of what values of \(N_{ij}\) are likely when we fit the model but leave out the count \(N_{ij}\). We then compare the true \(N_{ij}\) with this density and see how likely it is under the model we have chosen. More details of the method can be found in Gilks et al. (1996).

We carry out this cross-validation study in the following way. For specimen \(i\) we omit the count for polygon \(j\) and input all other data into the MCMC algorithm. All parameters \(\beta_1, \beta_2, \rho\) and \(\{\gamma_{ij}\}\) are estimated as usual. At each iteration \(r\) of the MCMC algorithm a count \(N_{ij}^{(r)}\) is simulated from a Poisson distribution where the parameter of the distribution is given by \(\mu_{ij}^{(r)} = (\beta_1^{(r)} C_j + \beta_2^{(r-1)} T_j + \sum_k \omega_{jk} \gamma_{ik}^{(r-1)} ) A_j\). For \(N_{ij}\) we then calculate the posterior predictive distribution as follows:

\[
P(N_{ij}|\{N_{-ij}\}) = \int P(N_{ij}|\theta, \{N_{-ij}\})P(\theta|\{N_{-ij}\})d\theta,
\]

where \(\theta = \beta_1, \beta_2, \rho, \{\gamma_{ij}\}\). We again use a Rao-Blackwellized estimate:

\[
\hat{P}(N_{ij}|\{N_{-ij}\}) = \frac{1}{R} \sum_{r=1}^{R} P(N_{ij}|\beta_1^{(r)}, \beta_2^{(r)}, \rho^{(r)}, \{\gamma_{ij}^{(r)}\}),
\]

where \(R\) is the total number of iterations after burn-in.

The results of this analysis are presented for Specimen 3 in Figure 11. Here we have omitted the count for each polygon in turn and predicted the count for the omitted polygon. As can be seen from the figure, all
true counts, except one, lie within the 95% predicted intervals, suggesting that the model does adequately fit the data. Another indication of the adequacy of the model is given by calculating the percentage of counts that fall into the 50% equal-tailed predictive intervals. This percentage is 47.7%, which suggests that the interval lengths are of the right size. We would expect with a reasonably well-fitting model, that approximately 50% of the observed data would lie in the predictive intervals.

6.3 Zero-Inflated Poisson Distribution

One point to note when examining Figure 10, as already mentioned in Section 6.1, is that the zero counts appear not to be so well modelled. The overabundance of observed zero counts could be modelled by treating the zero counts differently from the non-zero counts. One way of doing this is to consider a zero-inflated Poisson Distribution (Lambert (1992); Ridout et al. (1998)). $N$ has a zero-inflated Poisson (ZIP) distribution if

$$
P(N = n) = \begin{cases} 
\pi_0 + (1 - \pi_0) \exp(-\mu), & n = 0; \\
(1 - \pi_0) \frac{\exp(-\mu \mu^n)}{n!}, & n > 0,
\end{cases}
$$

where $\pi_0$ is the proportion of zero counts, $0 \leq \pi_0 < 1$. For the ZIP distribution

$$
E(N) = (1 - \pi_0) \mu; \\
\text{Var}(N) = (1 - \pi_0) \mu(1 + \pi_0 \mu).
$$

Thus the ZIP distribution has a variance that is greater than its mean, allowing for overdispersion.

We re-calculate the posterior predictive for the count of cracks in each polygon again using the Rao-Blackwellized estimator, Equation 9 and replace Equation 10 with the ZIP distribution, where $\pi_0$ is the proportion of polygons having zero count. For each of the polygons of Specimen 3 we present the actual count and the median posterior predicted count (and 95% quantiles), based on this estimate obtained using the ZIP distribution, see Figure 12. It would appear from this figure that the zero counts are better modelled using the ZIP distribution.

7 Discussion

In this analysis of crack location data in the bone cement of hip-replacement specimens we model the locations of the cracks as a heterogeneous Poisson process whose intensity is in turn modelled using an identity-link regression model, composed of crack initiating influential covariates, both measured (compression and tension) and unmeasured (pores, for example). Through a discrete latent spatial process we model the unknown influential factors, (such as pores) obtaining evidence of how influential these factors are together with the range over which they have an influence, the range being modelled through a Gaussian kernel. All parameter estimation is done using MCMC techniques, providing distributional estimates for all unknown parameters of interest. Apart from the zero counts, the model does seem to be a reasonable fit for the data as evidenced through our model validation studies. We suggest an alternative model to accommodate the zero counts, the ZIP distribution, and we show that this model is a better fit for the zero count data.

Choosing to model this type of data through a spatial process has not been considered before and so it is not possible to compare our results with those of other modellers. The results do agree with what is expected by the engineers, in particular the range over which the latent spatial factors have an influence. The suspicion that these factors would have a short-range effect is confirmed by our analysis, median of $\rho = 2.2379$, suggesting a range of influence of approximately 5 units. There does not appear, from the results of this analysis, to be a large difference in the contribution from the compression and tension stress as indicated by the parameters $\beta_1, \beta_2$. The collection of data on the distribution of pores is not currently feasible, but this analysis offers a means of identifying sources of influential factors that are crack causing but that have not been observed. Together with the identification of these factors we estimate their range of influence, again not measured in the laboratory experiment.

This model could be incorporated in a larger analysis examining the reliability if hip-replacement specimens, offering a spatial approach to such an analysis.
We have aggregated the data, leading to some loss of information as is inevitable when aggregation takes place. A continuous version of this model could be considered by treating the spatial process as a continuous process over the bone cement. This would lead to more accurate estimates of the unknown parameters but at the cost of a much more complicated model that would require more extensive analysis.

**Appendix A**

The joint posterior distribution for the augmented data may be written in the following form:

\[
\mathbb{P}(\beta_1, \beta_2, \{\gamma_{ij}\}, \rho|\{N_{ij}\}) \propto \exp \left\{ -\beta_1 \sum_{ij} C_j A_j \right\} (\beta_1)^{\sum_{ij} N_{ij1}} \times 5 \prod_j (C_j)^{N_{ij1}} \\
\times \exp \left\{ -\beta_2 \sum_{ij} T_j A_j \right\} (\beta_2)^{\sum_{ij} N_{ij2}} \times 5 \prod_j (T_j)^{N_{ij1}} \\
\times \prod_{ik} \left\{ \exp \left( -\beta_{ijk} \sum_j \omega_{jk} A_j \right) (\gamma_{ik})^{\sum_j N_{ijk+2}} \prod_j (\omega_{jk})^{N_{ijk+2}} \right\} \\
\times \exp \left\{ -b_1 \beta_1 / \beta_1^a - 1 \right\} \\
\times \exp \left\{ -b_2 \beta_2 / \beta_2^a - 1 \right\} \\
\times \frac{1}{\sigma} \exp \left\{ -\frac{(\log(\rho) - \mu)^2}{2\sigma^2} \right\} \\
\times \prod_{ik} \exp \left\{ -b_3 \gamma_{ik} \gamma_{ik}^{a_3 - 1} \right\}.
\]

The full conditional distributions for \(\beta_1, \beta_2, \) and \(\{\gamma_{ij}\}\) are obtained as follows:

\[
\mathbb{P}(\beta_1|\beta_2, \{\gamma_{ij}\}, \rho) \propto \exp \left\{ -\beta_1 \left( \sum_j C_j A_j + b_1 \right) \right\} \beta_1^{\sum_{ij} N_{ij1} + \alpha_1 - 1}, \\
\sim \text{Gamma} \left( \sum_{ij} N_{ij1} + \alpha_1, 5 \sum_j C_j A_j + b_1 \right);
\]

\[
\mathbb{P}(\beta_2|\beta_1, \{\gamma_{ij}\}, \rho) \propto \exp \left\{ -\beta_2 \left( \sum_j T_j A_j + b_2 \right) \right\} \beta_2^{\sum_{ij} N_{ij2} + \alpha_2 - 1}, \\
\sim \text{Gamma} \left( \sum_{ij} N_{ij2} + \alpha_2, 5 \sum_j T_j A_j + b_2 \right);
\]

\[
\mathbb{P}(\gamma_{ik}|\beta_1, \beta_2, \rho) \propto \exp \left\{ -\gamma_{ik} \left( \sum_j \omega_{jk} A_j + b_g \right) \right\} (\gamma_{ik})^{\sum_j N_{ijk+2} + \alpha_3 - 1}, \\
\sim \text{Gamma} \left( \sum_j N_{ijk+2} + \alpha_3, \sum_j \omega_{jk} A_j + b_g \right).
\]

The full conditional distribution for \(\rho\) is as follows:

\[
\mathbb{P}(\rho|\beta_1, \beta_2, \{\gamma_{ij}\}) \propto \prod_{ij} \left\{ \exp \left( -\sum_{jk} \gamma_{ik} \omega_{jk} A_j \right) (\lambda_{ij})^{N_{ij}} \right\} \frac{1}{\sigma \rho} \exp \left\{ -\frac{(\log(\rho) - \mu)^2}{2\sigma^2} \right\}.
\]

This distribution is not of known form, but instead can be used in a Metropolis random walk update for the parameter \(\rho\).
Figure 1: (a) Schematic diagram of a cross section of the hip replacement showing the bone cement encasing the prosthesis. (b) Exploded view of the laboratory model of the femoral stem and the bone cement. (Image courtesy of A.B. Lennon)

Figure 2: Kriged values of compression and tension for each of the polygons.
Figure 3: Diagrams on the left indicate individual crack locations for specimens 1 to 3 (top to bottom) with corresponding image diagrams on the right of the windows divided into an arbitrary grid of polygons shaded according to the observed number of cracks.
Figure 4: Diagrams on the left indicate individual crack locations for specimens 4 (top) and 5 (bottom) with corresponding image diagrams on the right of the windows divided into an arbitrary grid of polygons shaded according to the observed number of cracks.
Figure 5: Estimated log-likelihood, $\hat{p}(\{N_{ij}\}|\beta_1, \beta_2, \{\gamma_{ij}\}, \rho_m)$ calculated for $\rho_m \in [0.01, 20]$ for each of the five specimens. The log-likelihood was estimated using sample values for $\beta_1, \beta_2$, and $\{\gamma_{ij}\}$ from the posterior distribution obtained from the MCMC program. The dashed black line is at $\rho = 2$.

Figure 6: Lognormal densities for various mean and variance parameters.
Figure 7: Image plots for the simulation study, showing (a) stress, (b) $\gamma$’s, (c) simulated Poisson counts and (d) estimated $\gamma$’s.
Figure 8: The posterior median estimates of the latent contribution, $A_j \sum_k \omega_{jk} \gamma_{ik}$ for each polygon of each Specimen 1 to 5.
Figure 9: Posterior medians of $\lambda_{ij}$’s for Specimens 1 to 5.
Figure 10: For each polygon (x-axis) of Specimen 3 we show the actual count (red star), and the median posterior predicted count (black circle), together with 95% quantiles for the predicted counts.

Figure 11: For each polygon (x-axis) of Specimen 3 we show the actual count (red star), and the median posterior predicted count (black circle), together with 95% quantiles for the predicted counts, based on carrying out the analysis with the count for the relevant polygon omitted.
Figure 12: For each polygon (x-axis) of Specimen 3 we show the actual count (red star), and the median posterior predicted count (black circle), together with 95% quantiles for the predicted counts, based on the ZIP distribution.
Table 1: The first column indicates the parameter. The second column shows the actual values used in the simulation study and in column three the estimates (medians and (5%, 95%)) of these parameters obtained using the MCMC algorithm are shown. Column four shows estimates (medians and (5%, 95%)) of the parameters for the true experimental data obtained using the MCMC algorithm.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Simulated Values</th>
<th>MCMC Simulated Data</th>
<th>MCMC Experimental Data</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>0.01</td>
<td>0.0068 (0.00055, 0.0254)</td>
<td>0.0026 (0.00019, 0.0108)</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.03</td>
<td>0.01953 (0.00894, 0.0325)</td>
<td>0.0016 (0.00051, 0.0038)</td>
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<tr>
<td>$\rho$</td>
<td>3</td>
<td>3.138 (2.6816, 3.4777)</td>
<td>2.2379 (2.14006, 2.3349)</td>
</tr>
</tbody>
</table>
References


