Challenges to reasoning

Assaults against

- truth
  Liar’s Paradox: ‘I am lying’

- sets/membership ∈
  Russell set \( R = \{ x \mid \text{not } x \in x \} \)

- countability
  Cantor: \( \text{Power}(\{0, 1, 2, \ldots\}) \)

- change
  Sorites: heap (minus one grain)

- computability
  Turing: Halting Problem

The Halting Problem

Given a program \( P \) and data \( D \), return either 0 or 1 (as output), with 1 indicating that \( P \) halts on input \( D \)

\[
\text{HP}(P, D) := \begin{cases} 
1 & \text{if } P \text{ halts on } D \\
0 & \text{otherwise}
\end{cases}
\]

**Theorem (Turing)** No TM computes \( \text{HP} \).

The proof is similar to the Liar’s Paradox distributed as follows

\( H: \) ‘L speaks the truth’
\( L: \) ‘H lies’

with a spoiler L (exposing H as a fraud).
Proof of uncomputability

Given a TM $P$ that takes two arguments, we show $P$ does not compute HP by defining a TM $\overline{P}$ such that

$$P(\overline{P}, \overline{P}) \neq HP(\overline{P}, \overline{P}).$$

Let

$$\overline{P}(D) :\equiv \begin{cases} 1 & \text{if } P(D, D) = 0 \\ \text{loop} & \text{otherwise.} \end{cases}$$

and notice

$$HP(\overline{P}, \overline{P}) = \begin{cases} 1 & \text{if } \overline{P} \text{ halts on } \overline{P} \\ 0 & \text{otherwise} \end{cases} \quad \text{(def of HP)}$$

$$= \begin{cases} 1 & \text{if } P(\overline{P}, \overline{P}) = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{(def of } \overline{P})$$

\[\square\]

Semi-solvability of HP and reasoning

There is a TM that meets the positive part of HP (looping exactly when HP asks for 0), in view of the existence of a

**Universal Turing Machine**: a TM $U$ that runs $P$ on $D$

$$U(P, D) \sim P(D)$$

for any given TM $P$ and data $D$.

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**Negative results balanced by positive ways forward**

- deduction: Gödel incompleteness/completeness theorems
- truth: Tarski undefinability/definability